

Stephani-Schutz quantum cosmology

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Abstract

We study the Stephani quantum cosmological model in the presence of a cosmological constant in radiation dominated Universe. In the present work the Schutz's variational formalism which recovers the notion of time is applied. This gives rise to Wheeler-DeWitt equations which can be cast in the form of Schrödinger equations for the scale factor. We find their eigenvalues and eigenfunctions by using the Spectral Method. Then we use the eigenfunctions in order to construct wave packets and evaluate the time-dependent expectation value of the scale factor, which is found to oscillate between non-zero finite maximum and minimum values. Since the expectation value of the scale factor never tends to the singular point, we have an initial indication that this model may not have singularities at the quantum level.

Key words: Quantum cosmology, Stephani model,

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1. Introduction

In recent years observations show that the expansion of the Universe is accelerating in the present epoch [1] contrary to Friedmann-Robertson-Walker (FRW) cosmological models, with non-relativistic matter and radiation. Some different physical scenarios using exotic form of matter have been suggested to resolve this problem [2,3,4,5,6,7]. In fact the presence of exotic matter is not necessary to drive an accelerated expansion. Instead we can relax the assumption of the homogeneity of space, leaving the isotropy with respect to one point. The most general class of non-static, perfect fluid solutions of Einsteins equations that are conformally flat is known as the "Stephani Universe" [8,9]. This model can be embedded in a five-dimensional flat pseudo-Euclidean space, which is not expansion-free and has non-vanishing density [10,8,11]. In general, it has no symmetry at all, although its three dimensional spatial sections are homogeneous and isotropic [12]. The spherically symmetric Stephani Universes and some of their subcases have been examined in numerous papers [9]. So it may be important to study the quantum behavior of this model.

The notion of time can be recovered in some cases of quantum cosmology, for example when gravity is coupled to a perfect fluid [13,14,15]. This kind of systems are of-

ten studied as follows [7,14,15]. First one uses the Schutz's formalism for the description of the perfect fluid [16,17], second one selects the dynamical variable of perfect fluid as the reference time. Finally, one uses canonical quantization to obtain the Wheeler-Dewitt (WD) equation in minisuperspace, which is a Schrödinger-like equation [13]. After solving the equation, one can construct wave packets from the resulting modes. The wave packets can be used to compute the time-dependent behavior of the scale factor. If the selected time variable results in a close correspondence between the expectation value of the scale factor and the classical prediction (prediction of General Relativity) for long enough time, the selected time variable can be considered as acceptable. This approach has been extensively employed in the literature, indicating in general the suppression of the initial singularity [7,13,18,19,14,20,15].

In the present paper, we use the formalism of quantum cosmology in order to quantize the Stephani cosmological model in Schutz's formalism [16,17] and find WD equation in minisuperspace. In the Schutz's variational formalism the wave function depends on the scale factor a , and on the canonical variable associated to the fluid, which plays the role of time T . Here, we describe matter as a perfect fluid matter. Although, this is essentially semiclassical from the start, it has the advantage of defining a variable, connected with the matter degrees of freedom, which can naturally be identified with time and leads to a well-defined Hilbert space structure. Moreover, after the universe reaches the

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dust dominated matter, the evolution towards an exponentially expanding epoch involves a quantum mechanical transition associated with some vestige component of the original wave function of the Universe. In fact, the classical Universe on large scales is based on a quantum mechanical background. Particularly, a rapidly oscillating state with small amplitude, which is the cosmological influence of a wave function vestige, would emerge in the dust dominated epoch [26].

2. The Model

The action for gravity plus perfect fluid in Schutz's formalism is written as

$$S = \frac{1}{2} \int_M d^4x \sqrt{-g} (R - 2\Lambda) + 2 \int_{\partial M} d^3x \sqrt{h} h_{ab} K^{ab} + \int_M d^4x \sqrt{-g} p, \quad (1)$$

where K^{ab} is the extrinsic curvature, Λ is the cosmological constant, and h_{ab} is the induced metric over the three-dimensional spatial hypersurface, which is the boundary ∂M of the four dimensional manifold M in units where $8\pi G = 1$ [27]. The last term of (1) represents the matter contribution to the total action, and p is the pressure. In Schutz's formalism [16,17] the fluid's four-velocity is expressed in terms of five potentials ϵ , ζ , ξ , θ and S :

$$u_\nu = \frac{1}{\mu} (\epsilon_{,\nu} + \zeta \xi_{,\nu} + \theta S_{,\nu}), \quad (2)$$

where μ is the specific enthalpy, the variable S is the specific entropy, while the potentials ζ and ξ are connected with rotation and are absent in models of the FRW type. The variables ϵ and θ have no clear physical meaning. The four-velocity is subject to the normalization condition

$$u^\nu u_\nu = -1. \quad (3)$$

The metric in spherically symmetric Stephani Universe [10,28,8,12,9,29] has the following form,

$$ds^2 = - \left[F(t) \frac{a(t)}{V(r,t)} \frac{d}{dt} \left(\frac{V(r,t)}{a(r,t)} \right) \right]^2 dt^2 + \frac{a^2(t)}{V^2(r,t)} (dr^2 + r^2 d\Omega^2), \quad (4)$$

where the functions $V(r,t)$ and $F(t)$ are defined as

$$V(r,t) = 1 + \frac{1}{4} k(t) r^2, \quad (5)$$

$$F(t) = \frac{a(t)}{\sqrt{C^2(t) a^2(t) - k(t)}}. \quad (6)$$

Using the line element (4) and the Einstein's equation, one can easily show the functions $C(t)$, $k(t)$ and $a(t)$ are not all independent, but are related to each other with the following expressions

$$\rho(t) = \frac{3C^2(t) + \Lambda}{8\pi G}, \quad (7)$$

$$p(r,t) = \frac{1}{8\pi G} [2C(t)\dot{C}(t) \frac{V(r,t)/a(t)}{(V(r,t)/a(t))} - 3C^2(t) - \Lambda], \quad (8)$$

where an overdot denotes a derivative with respect to t . Note that in the spherically symmetric Stephani models and the given coordinate system, the energy density $\rho(t)$ is uniform, while the pressure $p(r,t)$ is not and depends on the distance from the symmetry center placed at $r = 0$. This is the reason why in such models the barotropic equation of state (i.e. of the form $p = p(\rho)$) does not exist. If, however, we assume some relations between $\rho(t)$ and $p(r,t)$, this could allow us to eliminate one of the unknown functions, e.g. $C(t)$. Hence we are left with two unknown functions $k(t)$ and $a(t)$. The first one $k(t)$ plays the role of a spatial curvature index, while the second one $a(t)$ is the Stephani version of the FRW scale factor.

Now we consider an observer placed at the symmetry center of the spherically symmetric Stephani Universe. All of our physical assumptions will concern the neighborhood $r \approx 0$. First of all we assume that locally, matter filling up the Universe fulfils a barotropic equation of state of the standard form

$$p(r \approx 0, t) = \alpha \rho(t). \quad (9)$$

By substituting the Stephani metric (4) in the action (1) and choosing a curvature function $k(t)$ in the form [30]

$$k(t) = \beta a^\gamma(t), \quad (10)$$

and after some thermodynamical considerations [13], the final reduced effective action near $r \approx 0$, takes the form

$$S = \int dt \left[-3 \frac{\dot{a}^2 a}{N} - \Lambda N a^3 + 3\beta N a^{1+\gamma} + N^{-1/\alpha} a^3 \frac{\alpha}{(\alpha+1)^{1/\alpha+1}} (\dot{\epsilon} + \theta \dot{S})^{1/\alpha+1} \exp\left(-\frac{S}{\alpha}\right) \right], \quad (11)$$

where $N = F(t)a(t)\frac{d}{dt}\left(\frac{1}{a(t)}\right)$. The reduced action may be further simplified by canonical methods [13] to the super-Hamiltonian

$$\mathcal{H} = -\frac{p_a^2}{12a} + \Lambda a^3 - 3\beta a^{1+\gamma} + \frac{p_\epsilon^{\alpha+1} e^S}{a^{3\alpha}}, \quad (12)$$

where $p_a = -6\dot{a}a/N$ and $p_\epsilon = -\rho_0 u^0 N a^3$, ρ_0 being the rest mass density of the fluid. Using the canonical transformations

$$T = p_S e^{-S} p_\epsilon^{-(\alpha+1)}, \quad p_T = p_\epsilon^{\alpha+1} e^S, \\ \bar{\epsilon} = \epsilon - (\alpha+1) \frac{p_S}{p_\epsilon}, \quad \bar{p}_\epsilon = p_\epsilon, \quad (13)$$

which are the generalization of the ones used in Ref. [13], the super-Hamiltonian takes the form

$$\mathcal{H} = -\frac{p_a^2}{12a} + \Lambda a^3 - 3\beta a^{1+\gamma} + \frac{p_T}{a^{3\alpha}}, \quad (14)$$

where the momentum p_T is the only remaining canonical variable associated with matter and appears linearly in the super-Hamiltonian.

The classical dynamics is governed by the Hamilton equations, derived from Eq. (14) and Poisson brackets, namely

$$\left\{ \begin{array}{l} \dot{a} = \{a, N\mathcal{H}\} = -\frac{Np_a}{6a}, \\ \dot{p}_a = \{p_a, N\mathcal{H}\} = -\frac{N}{12a^2}p_a^2 + 3N(1+\gamma)\beta a^\gamma \\ \quad - 3N\Lambda a^2 + \frac{3N\alpha}{a^{1+3\alpha}}p_T, \\ \dot{T} = \{T, N\mathcal{H}\} = Na^{-3\alpha}, \\ \dot{p}_T = \{p_T, N\mathcal{H}\} = 0. \end{array} \right. \quad (15)$$

We also have the constraint equation $\mathcal{H} = 0$. Choosing the gauge $N = a^{3\alpha}$, we have the following equations for the system

$$T = t, \quad (16)$$

$$\ddot{a} = (3\alpha - \frac{1}{2})\frac{\dot{a}^2}{a} - \frac{1}{2}(1+\gamma)\beta a^{6\alpha+\gamma-1} + \frac{1}{2}\Lambda a^{6\alpha+1} - \frac{\alpha}{2}a^{3\alpha-2}p_T, \quad (17)$$

$$0 = -\frac{3\dot{a}^2}{a^{6\alpha-1}} - 3\beta a^{\gamma+1} + \Lambda a^3 + \frac{p_T}{a^{3\alpha}}. \quad (18)$$

Note that the classical equations for the case $\gamma = +1$, in Ref. [31], correspond with choosing $\Lambda = 0$ and $N = 1$. In this case ($\Lambda = 0$, $N = 1$) the constraint equation $\mathcal{H} = 0$ reduces to

$$-3a\dot{a}^2 - 3\beta a^2 + a^{-3\alpha}p_T = 0, \quad (19)$$

or

$$\left(\frac{da(t)}{dt}\right)^2 + \beta a(t) = \frac{p_T}{3a^{3\alpha+1}(t)}. \quad (20)$$

Imposing the standard quantization conditions on the canonical variables ($p_a \rightarrow -i\frac{\partial}{\partial a}$, $p_T \rightarrow -i\frac{\partial}{\partial T}$) and demanding that the super-Hamiltonian operator (14) annihilate the wave function, we are led to the following WD equation in minisuperspace ($\hbar = 1$)

$$\frac{\partial^2 \Psi}{\partial a^2} + (12\Lambda a^4 - 36\beta a^{2+\gamma})\Psi - i12a^{1-3\alpha}\frac{\partial \Psi}{\partial t} = 0. \quad (21)$$

According to the equation (16) $T = t$, can be associated with the time coordinate [32,33]. Equation (21) takes the form of a Schrödinger equation $i\partial\Psi/\partial t = \hat{H}\Psi$. As discussed in [19,33], in order for the Hamiltonian operator \hat{H} to be

self-adjoint the inner product of any two wave functions Φ and Ψ must take the form

$$(\Phi, \Psi) = \int_0^\infty a^{1-3\alpha}\Phi^*\Psi da. \quad (22)$$

Moreover, the wave functions should satisfy the restrictive boundary conditions which the simplest ones are [33,34]

$$\Psi(0, t) = 0 \quad \text{or} \quad \left.\frac{\partial \Psi(a, t)}{\partial a}\right|_{a=0} = 0. \quad (23)$$

The WD equation (21) can be solved by separation of variables as follows,

$$\Psi(a, t) = e^{iE_n t}\psi(a), \quad (24)$$

where the scale factor dependent part of the wave function $\psi(a)$ satisfies

$$-\psi'' + (36\beta a^{2+\gamma} - 12\Lambda a^4)\psi = 12E_n a^{1-3\alpha}\psi, \quad (25)$$

and the prime denotes derivative with respect to a . The interesting feature of the Stephani model is that the spatial curvature is time-dependent. The recent observational data shows that our Universe is spatially flat. Moreover, negative powers in equation (10) lead to the spatially flat Universe in the present epoch.

We construct a general solution to the WD equation (21) by taking linear combinations of the $\psi_n(a)$'s,

$$\Psi(a, t) = \sum_{n=0}^m C_n \psi_n(a) e^{iE_n t}, \quad (26)$$

where the coefficients $C_n(E_n)$ will be fixed later by choosing appropriate boundary conditions. we can compute the expected value for the scale factor a for arbitrary wave packets, following the *many worlds interpretation* of quantum mechanics [35,36]. We may write the expected value for the scale factor a , with regards to the inner product (22) as

$$\langle a \rangle_t = \frac{\int_0^\infty a^{2-3\alpha} |\Psi(a, t)|^2 da}{\int_0^\infty a^{1-3\alpha} |\Psi(a, t)|^2 da}. \quad (27)$$

3. The Spectral Method

To solve the resulting WD equation (25) we use Spectral Method (SM) [37,38,39,40] for finding the bound states of (25). SM is simple, fast, accurate, robust and stable. This method consists of first, choosing a finite domain for the approximate solution denoted by $2L$, and second taking the solution as a finite superposition of the Fourier basis functions in this domain which satisfy the periodic boundary condition. By substituting the expansion into the differential equation a matrix equation is obtained. By minimizing the first eigenvalue of the resulting matrix with respect to the value of spatial domain $2L$, the optimized basis functions can be found. The accurate energy eigenvalues correspond to the eigenvalues of the resulting matrix with optimized basis functions. We only examine the bound states

of this problem, i.e. the states which are the square integrable. The general equation that we want to solve can be written in the form

$$f_1(x) \frac{d^2 \psi(x)}{dx^2} + f_2(x) \frac{d \psi(x)}{dx} + f_3(x) \psi(x) = \varepsilon \psi(x), \quad (28)$$

Any complete orthonormal set can be used for the SM. We use the Fourier series basis. That is, since we need to choose a finite subspace of a countably infinite basis, we restrict ourselves to the finite region $-L < x < L$. The value of L is determined by requiring that the sought-after eigenfunctions have compact support in this domain subject to the aforementioned optimization. This means that we can expand the solution as

$$\psi(x) = \sum_{i=1}^2 \sum_{m=0}^{\infty} A_{m,i} u_i \left(\frac{m\pi x}{L} \right), \quad (29)$$

where

$$\begin{cases} u_1 \left(\frac{m\pi x}{L} \right) = \frac{1}{\sqrt{LR_m}} \sin \left(\frac{m\pi x}{L} \right), \\ u_2 \left(\frac{m\pi x}{L} \right) = \frac{1}{\sqrt{LR_m}} \cos \left(\frac{m\pi x}{L} \right), \end{cases} \quad R_m = \begin{cases} 2, & m=0, \\ 1, & \text{otherwise.} \end{cases} \quad (30)$$

That is we assume periodic boundary condition. We can also make the following expansions

$$f_1(x) \frac{d^2 \psi(x)}{dx^2} = \sum_{m,i} B_{m,i} u_i \left(\frac{m\pi x}{L} \right), \quad (31)$$

$$f_2(x) \frac{d \psi(x)}{dx} = \sum_{m,i} C_{m,i} u_i \left(\frac{m\pi x}{L} \right), \quad (32)$$

$$f_3(x) \psi(x) = \sum_{m,i} D_{m,i} u_i \left(\frac{m\pi x}{L} \right), \quad (33)$$

where $B_{m,i}$, $C_{m,i}$ and $D_{m,i}$ are coefficients that can be determined once $f_1(x)$, $f_2(x)$ and $f_3(x)$ are specified. By substituting and using the differential equation of the Fourier basis we obtain

$$\sum_{m,i} \left[B_{m,i} + C_{m,i} + D_{m,i} \right] u_i \left(\frac{m\pi x}{L} \right) = \varepsilon \sum_{m,i} A_{m,i} u_i \left(\frac{m\pi x}{L} \right). \quad (34)$$

Because of the linear independence of $u_i \left(\frac{m\pi x}{L} \right)$, every term in the summation must satisfy

$$B_{m,i} + C_{m,i} + D_{m,i} = \varepsilon A_{m,i}. \quad (35)$$

It only remains to determine the matrices B , C and D . Using Eq. (29) and Eqs. (31,32,33) we have

$$\sum_{m,i} B_{m,i} u_i \left(\frac{m\pi x}{L} \right) = - \sum_{m,i} A_{m,i} \left(\frac{m\pi}{L} \right)^2 f_1(x) u_i \left(\frac{m\pi x}{L} \right), \quad (36)$$

$$\sum_{m,i} C_{m,i} u_i \left(\frac{m\pi x}{L} \right) = \sum_{m,i} A_{m,i} f_2(x) \frac{d}{dx} u_i \left(\frac{m\pi x}{L} \right), \quad (37)$$

$$\sum_{m,i} D_{m,i} u_i \left(\frac{m\pi x}{L} \right) = \sum_{m,i} A_{m,i} f_3(x) u_i \left(\frac{m\pi x}{L} \right). \quad (38)$$

By multiplying both sides of the above equations by $u_{i'} \left(\frac{m'\pi x}{L} \right)$ and integrating over the x -space and using the orthonormality condition of the basis functions, one finds

$$\begin{aligned} B_{m,i} &= - \sum_{m',i'} A_{m',i'} \left(\frac{m\pi}{L} \right)^2 \int_{-L}^L u_i \left(\frac{m\pi x}{L} \right) f_1(x) u_{i'} \left(\frac{m'\pi x}{L} \right) dx, \\ &= \sum_{m',i'} b_{m,m',i,i'} A_{m',i'}, \end{aligned} \quad (39)$$

$$\begin{aligned} C_{m,i} &= \sum_{m',i'} A_{m',i'} \int_{-L}^L u_i \left(\frac{m\pi x}{L} \right) f_2(x) \frac{d}{dx} u_{i'} \left(\frac{m'\pi x}{L} \right) dx, \\ &= \sum_{m',i'} c_{m,m',i,i'} A_{m',i'}, \end{aligned} \quad (40)$$

$$\begin{aligned} D_{m,i} &= \sum_{m',i'} A_{m',i'} \int_{-L}^L u_i \left(\frac{m\pi x}{L} \right) f_3(x) u_{i'} \left(\frac{m'\pi x}{L} \right) dx, \\ &= \sum_{m',i'} d_{m,m',i,i'} A_{m',i'}. \end{aligned} \quad (41)$$

Therefore we can rewrite Eq. (35) as

$$\sum_{m',i'} \left[b_{m,m',i,i'} + c_{m,m',i,i'} + d_{m,m',i,i'} \right] A_{m',i'} = \varepsilon A_{m,i}. \quad (42)$$

By selecting a finite subset of the basis functions, *e.g.* choosing the first $2N$ which could be accomplished by letting the index m run from 1 to N in the summations, equation (42) can be written as

$$D A = \varepsilon A, \quad (43)$$

where D is a square matrix with $(2N) \times (2N)$ elements. Its elements can be obtained from Eq. (42). The eigenvalues and eigenfunctions of the Schrödinger equation are approximately equal to the corresponding quantities of the matrix D . That is the solution to this matrix equation simultaneously yields $2N$ sought after eigenstates and eigenvalues.

4. Results

Since the Hamiltonian commutes with Parity operator, eigenstates divide into odd and even categories. We choose the odd solutions which satisfy the first boundary condition (23). We are free to choose the parameters β , γ , and Λ . Unlike FRW models in which the bound states merely correspond to negative cosmological constants [34], in this model we can find the bound states with negative or positive cosmological constant by choosing suitable value of γ . In radiation regime ($\alpha = 1/3$), comparing equations (25) and (28) we have

$$f_1(x) = -1, \quad f_2(x) = 0, \quad (44)$$

$$f_3(x) = 36\beta a^{\gamma+2} - 12\Lambda a^4, \quad \varepsilon = 12E. \quad (45)$$

Here, we restrict ourselves to two cases: $(\beta = 1, \gamma = 4, \Lambda = 1)$ and $(\beta = 1, \gamma = -2, \Lambda = -1)$. In radiation dominated regime the expectation value of the scale factor can be written as (27)

$$\langle a \rangle_t = \frac{\int_0^\infty a |\Psi(a, t)|^2 da}{\int_0^\infty |\Psi(a, t)|^2 da}. \quad (46)$$

Table 1 shows the first 20 odd energy eigenvalues of (25) for the two mentioned categories with 10 significant digits. We can now construct the wave packets (26) by superimposing the resulting eigenfunctions (26). Here, we choose first 20 eigenfunctions ($m = 20$) and, for simplicity according to Ref. [34], select the coefficients equal to one ($C_n = 1$) to incorporate equally all energy levels. As can be seen from the classical equations of motion (17,18), the model has singularities at the classical level. At the quantum level the probability density of finding the scale factor is (27)

$$P(a, t) = a^{1-3\alpha} |\Psi(a, t)|^2. \quad (47)$$

Since the probability density of finding the scale factor at $a = 0$ in radiation regime is zero for odd solutions at all times ($\lim_{a \rightarrow 0} |\Psi(a, t)|^2 = 0$), we have a initial indication that these models may not have singularities at the quantum level. Figures 1 and 2 show the behavior of expectation value of the scale factor for two mentioned cases in comparison with the classical behavior. Although, in classical case the scale factors reach the zero axes, the expectation value of the scale factors never tend to the singular point. This is depicted in Figs. (3,4) which show the long time behavior of the scale factors. This means that the big bang and big crunch phenomena are absent at quantum level. Similar properties have been discussed in [13,34,7] for FRW cosmological models.

5. Conclusion

In this work we have investigated perfect fluid Stephani quantum cosmological model in the presence of cosmological constant. The use of Schutz's formalism allows us to obtain a Schrödinger-like WD equation in which the only remaining matter degree of freedom plays the role of time. We have obtained eigenfunctions and therefore acceptable wave packets have been constructed by appropriate linear combination of these eigenfunctions. The time evolution of the expectation value of the scale factor has been determined in the spirit of the many-worlds interpretation of quantum cosmology. We have shown that, contrary to the classical case, the expectation values of the scale factor avoid singularity in the quantum case.

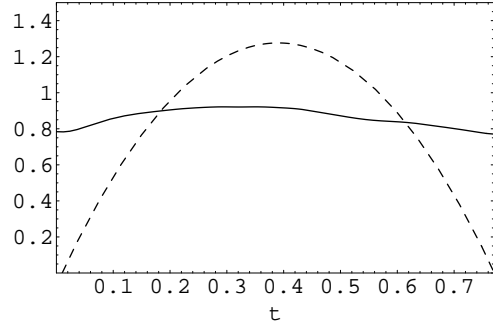


Fig. 1. Classical behavior of the scale factor (dashed line) and the quantum mechanical expectation value of the scalar factor (solid line) for $\beta = 1$, $\Lambda = 1$, and $\gamma = 4$ in radiation regime.

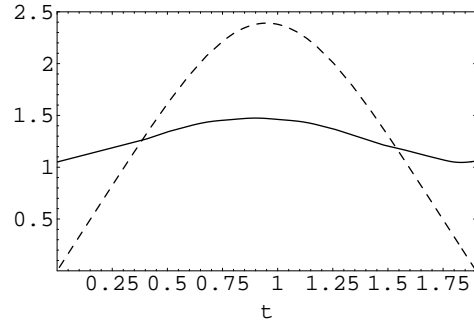


Fig. 2. Classical behavior of the scale factor (dashed line) and the quantum mechanical expectation value of the scalar factor (solid line) for $\beta = 1$, $\Lambda = -1$, and $\gamma = -2$ in radiation regime.

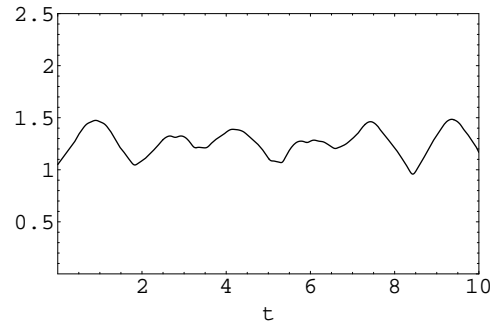


Fig. 3. The expectation value of the scalar factor for $\beta = 1$, $\Lambda = -1$, and $\gamma = -2$ in radiation regime for a long period of time.

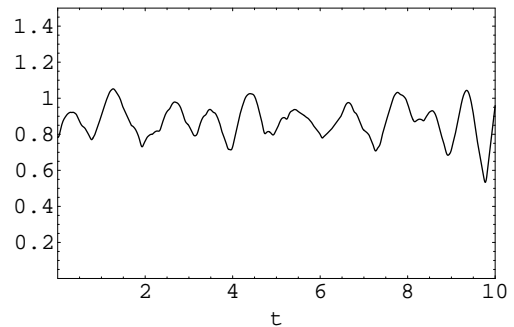


Fig. 4. The expectation value of the scalar factor for $\beta = 1$, $\Lambda = 1$, and $\gamma = 4$ in radiation regime for a long period of time.

	$\beta = 1, \gamma = 4, \Lambda = 1$	$\beta = 1, \gamma = -2, \Lambda = -1$
E_1	0.7404299830	3.724923306
E_2	2.716898545	5.221651006
E_3	5.467397344	7.051977995
E_4	8.816858136	9.123953661
E_5	12.67642129	11.39097635
E_6	16.98807771	13.82407714
E_7	21.71008635	16.40315278
E_8	26.81052795	19.11327742
E_9	32.26395981	21.94284703
E_{10}	38.04947945	24.88253248
E_{11}	44.14951426	27.92463946
E_{12}	50.54902037	31.06269279
E_{13}	57.23492813	34.29115371
E_{14}	64.19574440	37.60522022
E_{15}	71.42125871	41.00068165
E_{16}	78.90232098	44.47381007
E_{17}	86.63066966	48.02127735
E_{18}	94.59879650	51.64009065
E_{19}	102.7998386	55.32754141
E_{20}	111.2274909	59.08116446

Table 1

The lowest calculated odd energy levels for two cases in radiation dominated Universe.

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